

Toroidal Soliton Solutions in $O(3)^N$ Nonlinear Sigma Model.

A. Wereszczyński ^{a)} *

^{a)} Institute of Physics, Jagiellonian University,
Reymonta 4, Kraków, Poland

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Abstract

A set of N three component unit scalar fields in $(3+1)$ Minkowski space-time is investigated. The highly nonlinear coupling between them is chosen to omit the scaling instabilities. The multi-soliton static configurations with arbitrary Hopf numbers are found. Moreover, the generalized version of the Vakulenko-Kapitansky inequality is obtained. The possibility of attractive as well as repulsive interaction between hopfions is shown. A noninteracting limit is also discussed.

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*wereszcz@alphas.if.uj.edu.pl

1 Introduction

Topological defects are one of the main ideas of the temporary theoretical physics. They play crucial role in many models - from cosmological strings and textures [1], [2] to magnetic monopoles and vortices in Quantum Chromodynamics [3], [4]. On the other hand, topological defects are observed in various condense matter experiments - see for instance liquid crystals and ^3He [5] or ^4He quantum liquids [6]. Among quite well understood topological defects like domain walls, strings/vortices and monopoles toroidal-like configurations, that is solitons with nonzero value of the Hopf index, seem to be rather mysterious objects.

Since Faddeev, Niemi [7], [8] and Cho [9] have proposed their famous effective model for the low energy gluodynamics, where glueballs (particles which consist of only the gauge fields) are knot-like objects made of self-linking flux-tubes, toroidal solitons, their physical as well as mathematical aspects have been considered by many theoreticians ¹. Unfortunately, due to the rather complicated toroidal symmetry and nonlinearity of equations of motion no analytical results, in recently investigated QCD relevant models [12], [13], [14], have been found. Almost all results have been obtained by means of numerical methods [15], [16], [17], [18]. In order to deal with exact toroidal solutions one is forced to consider various toy models.

All such theories, providing analytical description of hopfions, are based on the quite old idea presented by Deser et al. [19]. They have investigated a classical highly nonlinear field theory with a form of the Lagrangian chosen to circumvent Derrick's theorem. In fact a soliton solution with unit Hopf index [20] and then configurations with arbitrary value of the Hopf invariant have been found [21]. Moreover, nontrivial topological solutions have been obtained in more complicated model [22], for example with explicitly broken $O(3)$ symmetry [23]. Using the exact soliton solutions one has checked that the Vakulenko-Kapitanski [24] inequality is fulfilled in all these models.

It should be mentioned that soliton solutions with non-trivial Hopf index can be achieved also in another famous model i.e. Skyrme model in $(3 + 1)$ dimensions [25], [26]. Moreover, there is also possibility to analyze hopfions in less dimensional space-time. It is based on the observation that the Faddeev-Niemi knots can be understood as twisted magnetic vortex rings built of so-called new baby skyrmions found in $(2 + 1)$ anisotropic Skyrme model [27].

In the present paper we would like to investigate toroidal soliton solutions in

¹Knotted solitons appear also in condensed matter [10] and possibly in astrophysics [11].

a model which consists of many unit vector fields i.e. allows for many topological charges. The appearance of more than one unit vector field has been observed in recently derived non-Abelian generalization of the color dielectric model [14]. Indeed, there are two unit fields in this model. The main aim, besides finding exact form of the topological solutions and their energies and Hopf charges, is to analyze Vakulenko-Kapitanski formula in case of more than one Hopf index.

2 The model

Let us consider a set of N unit three component scalar fields \vec{n}_i , $i = 1 \dots N$, $\vec{n}_i^2 = 1$, living in the $(3 + 1)$ Minkowski space-time. The Lorentz and $O(3)^N$ invariant Lagrangian density is chosen in the following form

$$\mathcal{L} = \prod_{i=1}^N \left([\vec{n}_i \cdot (\partial_\mu \vec{n}_i \times \partial_\nu \vec{n}_i)]^2 \right)^{\alpha_i} \quad (1)$$

which is a generalized version of the Aratyn-Ferreira-Zimmerman model with N unit fields. It can be rewritten as $\mathcal{L} = \prod_{i=1}^N (H_{\mu\nu}^{(i)})^{\alpha_i}$, where $H_{\mu\nu}^{(i)} = \vec{n}_i \cdot (\partial_\mu \vec{n}_i \times \partial_\nu \vec{n}_i)$ are antisymmetric field tensors. In order to omit the Derrick arguments for nonexistence of static soliton solutions one has to assume the condition

$$\sum_{i=1}^N \alpha_i = \frac{3}{4}. \quad (2)$$

In fact, now the total energy becomes invariant under the scale transformations.

Now, after taking advantage of the stereographic projection, we can express each of the unit fields in terms of two complex functions u and u^* :

$$\vec{n}_i = \frac{1}{1 + |u_i|^2} (u_i + u_i^*, -i(u_i - u_i^*), |u_i|^2 - 1). \quad (3)$$

Then the Lagrangian density reads

$$\mathcal{L} = 8^{3/4} \prod_{i=1}^N \frac{1}{(1 + |u_i|^2)^{4\alpha_i}} (K_\mu^{(i)} \partial^\mu u_i^*)^{\alpha_i}, \quad (4)$$

where we have introduced a set of the objects:

$$K_\mu^{(i)} = (\partial_\nu u_i^* \partial^\nu u_i) \partial_\mu u_i - (\partial_\nu u_i \partial^\nu u_i) \partial_\mu u_i^*, \quad (5)$$

where $i = 1 \dots N$. It is straightforward to check that they fulfill the following conditions

$$K_\mu^{(i)} \partial^\mu u_i = 0 \quad \text{and} \quad \text{Im}(K_\mu^{(i)} \partial^\mu u_i^*) = 0. \quad (6)$$

The equations of motion take the form

$$\begin{aligned} & \partial_\mu \left[\left(\prod_{i=1, i \neq j}^N \frac{(K_\mu^{(i)} \partial^\mu u_i^*)^{\alpha_i}}{(1 + |u_i|^2)^{4\alpha_i}} \right) \frac{2\alpha_j (K_\mu^{(j)} \partial^\mu u_j^*)^{\alpha_j-1}}{(1 + |u_j|^2)^{4\alpha_j}} K^{(j)\mu} \right] - \\ & - \prod_{i=1, i \neq j}^N \frac{(K_\mu^{(i)} \partial^\mu u_i^*)^{\alpha_i}}{(1 + |u_i|^2)^{4\alpha_i}} (K_\mu^{(j)} \partial^\mu u_j^*)^{\alpha_j} \left[\frac{1}{(1 + |u_j|^2)^{4\alpha_j}} \right]_{u_j^*}' = 0. \end{aligned} \quad (7)$$

After some calculation one can rewrite them as follow

$$\partial_\mu \left[\left(\prod_{i=1, i \neq j}^N \frac{(K_\mu^{(i)} \partial^\mu u_i^*)^{\alpha_i}}{(1 + |u_i|^2)^{4\alpha_i}} \right) \frac{(K_\mu^{(j)} \partial^\mu u_j^*)^{\alpha_j-1}}{(1 + |u_j|^2)^{4\alpha_j-2}} K^{(j)\mu} \right] = 0, \quad (8)$$

or in the more compact form

$$\partial_\mu \mathcal{K}^{(j)\mu} = 0, \quad (9)$$

where

$$\mathcal{K}_\mu^{(j)} = \left(\prod_{i=1, i \neq j}^N \frac{(K_\mu^{(i)} \partial^\mu u_i^*)^{\alpha_i}}{(1 + |u_i|^2)^{4\alpha_i}} \right) \frac{(K_\mu^{(j)} \partial^\mu u_j^*)^{\alpha_j-1}}{(1 + |u_j|^2)^{4\alpha_j-2}} K^{(j)\mu}. \quad (10)$$

It is easy to show that these quantities also fulfill the previously found conditions (6). Because of that we can generalize the procedure defined in [21] and construct the infinity families of the conserved currents

$$\mathcal{J}_\mu^{(i)} = \mathcal{K}^{(i)\mu} \frac{\partial G_i}{\partial u_i} - \mathcal{K}^{*(i)\mu} \frac{\partial G_i}{\partial u_i^*}, \quad (11)$$

where G_i , $i = 1 \dots N$ are any functions of u_j and u_j^* , $j = 1 \dots N$. Due to that presented $O(3)^N$ invariant model is integrable in the sense that N infinite families of the conserved currents can be found [28], [29].

Let us proceed further and prove that, as in case of soliton models in $(1+1)$ and $(2+1)$ dimension, the existence of infinite number of the conserved currents leads to topological soliton solutions. We begin with introducing of the toroidal variables

$$\begin{aligned} x &= \frac{a}{q} \sinh \eta \cos \phi, \\ y &= \frac{a}{q} \sinh \eta \sin \phi, \end{aligned}$$

$$z = \frac{a}{q} \sin \xi, \quad (12)$$

where $q = \cosh \eta - \cos \xi$ and the constant $a > 0$ sets the scale of the coordinates. However, one should keep in mind that the model is scale invariant. Due to that, there is no scale on the solution level. All solitons, irrespective of size, have identical energy.

In addition we assume a natural generalization of the Aratyn-Feriera-Zimmerman Ansatz [21]

$$u_i(\eta, \xi, \phi) \equiv f_i(\eta) e^{i(m_i \xi + n_i \phi)}, \quad i = 1 \dots N, \quad (13)$$

where m_i, n_i , $i = 1 \dots N$ are integral parameters.

Inserting the Ansatz into the field equations (9) we derive at the following static equations

$$\begin{aligned} \partial_\eta \ln \left[\left(\prod_{i=1, i \neq j}^N \frac{(f_i f'_i)^{2\alpha_i}}{(1 + f_i^2)^{4\alpha_i}} \right) \frac{(f_j f'_j)^{2\alpha_j - 1}}{(1 + f_j^2)^{4\alpha_j - 2}} \right] = \\ = -\partial_\eta \ln \left[\sinh \eta \prod_{i=1}^N \left(m_i^2 + \frac{n_i^2}{\sinh^2 \eta} \right)^{\alpha_i} \right]. \end{aligned} \quad (14)$$

These equations can be integrate and we obtain that

$$\left(\prod_{i=1, i \neq j}^N \frac{(f_i f'_i)^{2\alpha_i}}{(1 + f_i^2)^{4\alpha_i}} \right) \frac{(f_j f'_j)^{2\alpha_j - 1}}{(1 + f_j^2)^{4\alpha_j - 2}} = \frac{k_j}{\sinh \eta} \prod_{i=1}^N \left(m_i^2 + \frac{n_i^2}{\sinh^2 \eta} \right)^{-\alpha_i}, \quad (15)$$

where the integration constants k_j , $j = 1 \dots N$ have been introduced.

One can show that (15) can be reduced to the set of N decoupled first order differential equations

$$\frac{f_j f'_j}{(1 + f_j^2)^2} = \frac{1}{k_j} \left(\prod_{i=1}^N k_i^{4\alpha_i} \left(m_i^2 + \frac{n_i^2}{\sinh^2 \eta} \right)^{-2\alpha_i} \right) \frac{1}{\sinh^2 \eta}, \quad (16)$$

with the following general solutions

$$\frac{1}{1 + f_j^2} = \frac{1}{k_j} \int \left(\prod_{i=1}^N k_i^{4\alpha_i} \left(m_i^2 + \frac{n_i^2}{\sinh^2 \eta} \right)^{-2\alpha_i} \right) \frac{d\eta}{\sinh^2 \eta} + l_j. \quad (17)$$

Here l_j , $j = 1 \dots N$ are integration constants.

Now, we will analyze the total energy corresponding to our model (1):

$$E \equiv \int d^3 x T_{00} = 8^{3/4} \int d^3 x \prod_{i=1}^N \frac{(K_j^{(i)} \partial^j u_i^*)^{\alpha_i}}{(1 + |u_i|^2)^{4\alpha_i}}, \quad (18)$$

where the stereographic projection (3) has been taken into account. Inserting (5) and the Ansatz (13) into (18) we derive that

$$E_{m,n} = (2\pi)^2 8 \cdot 2^{3/4} \int_0^\infty d\eta \sinh \eta \left(\prod_{i=1}^N \frac{(f_i f'_i)^{2\alpha_i}}{(1 + f_i^2)^{4\alpha_i}} \left(m_i^2 + \frac{n_i^2}{\sinh^2 \eta} \right)^{\alpha_i} \right) \quad (19)$$

Quite surprisingly, using the equations (16) we are able to remove the unknown functions f_j from this formula. Then the total energy integral takes the form

$$E_{m,n} = (2\pi)^2 8 \cdot 2^{3/4} \prod_{i=1}^N k_i^{4\alpha_i} \int_0^\infty \frac{d\eta}{\sinh^2 \eta} \prod_{i=1}^N \left(m_i^2 + \frac{n_i^2}{\sinh^2 \eta} \right)^{-2\alpha_i}, \quad (20)$$

which can be evaluated, at least by means of some numerical methods, for all values of the parameters m_i, n_i, α_i .

One can notice that behavior of the integral (and difficulty of the calculation) strongly depends on a particular value of the following ration

$$\frac{n_i^2}{m_i^2} = q_i^2. \quad (21)$$

In the next section the simplest case, when all these numbers are equal will be investigated.

3 Case $q^2 = \text{const.}$

Let us focus on the case when

$$\frac{n_i^2}{m_i^2} = q^2 = \text{const.}, \quad (22)$$

for all $i = 1 \dots N$. Then the equations of motion (16) can be simplified to the following expressions

$$\frac{f_j f'_j}{(1 + f_j^2)^2} = \frac{1}{k_j} \prod_{i=1}^N k_i^{4\alpha_i} \left(m_i^2 + \frac{n_i^2}{\sinh^2 \eta} \right)^{-2\alpha_i} \frac{1}{\sinh^2 \eta}. \quad (23)$$

Moreover, these equations can be integrated out and finally we obtain that

$$\frac{1}{1 + f_j^2} = \frac{1}{k_j} \frac{1}{1 - q^2} \left(\prod_{i=1}^N k_i^{4\alpha_i} m_i^{-4\alpha_i} \right) \frac{\cosh \eta}{(q^2 + \sinh^2 \eta)^{\frac{1}{2}}} + l_j, \quad (24)$$

where $j = 1 \dots N$. Here we have taken $q^2 < 1$. In order to find the value of the integration constants we need to specify the asymptotical conditions for the functions f_j . We choose them in the form which admits the nontrivial topological structure [21]

$$\vec{n} \rightarrow (0, 0, 1) \text{ i.e. } f \rightarrow \infty \text{ as } \eta \rightarrow 0 \quad (25)$$

and

$$\vec{n} \rightarrow (0, 0, -1) \text{ i.e. } f \rightarrow 0 \text{ as } \eta \rightarrow \infty. \quad (26)$$

Then we find that

$$\frac{\prod_{i=1}^N k_i^{4\alpha_i}}{k_j} = \frac{1}{2 \prod_{i=1}^N m^{-4\alpha_i}} \frac{1 - q^2}{|q| - 1} |q|. \quad (27)$$

This is equivalent to the fact that all integration constants have to take the same value

$$k_j = k = \text{const.} \quad (28)$$

In addition, one can obtain that

$$l_j = l = -\frac{1}{|q| - 1}. \quad (29)$$

Finally, the solution is given by the following formula

$$\frac{1}{1 + f_j^2} = \frac{1}{1 - |q|} \left[1 - |q| \frac{\cosh \eta}{q^2 + \sinh^2 \eta} \right]. \quad (30)$$

One can recognize here the set of N Aratyn-Ferreira-Zimmerman toroidal solutions.

Let us now show that the solution possesses toroidal symmetry. The components of the unit vector fields read

$$n_{(i)}^1 = \frac{2f_i}{1 + f_i^2} \cos(m_i \xi + n_i \phi), \quad (31)$$

$$n_{(i)}^2 = \frac{2f_i}{1 + f_i^2} \sin(m_i \xi + n_i \phi), \quad (32)$$

$$n_{(i)}^3 = \frac{-1 + f_i^2}{1 + f_i^2}, \quad (33)$$

where f_i is given by (30) i.e. depends only on the radial-like coordinate η . It is clearly visible that surfaces of constant $n_{(i)}^3$ are torii.

After inserting obtained here solution into the total energy integral one finds that

$$E_{m,n} = (2\pi)^2 4 \cdot 2^{1/4} \prod_{i=1}^N m_i^{2\alpha_i} \sqrt{1+|q|} \sqrt{|q|}. \quad (34)$$

In the limit $m_{\alpha_i} = m$, $i = 1 \dots N$ the formula founded in [21] can be reproduced.

In order to prove that our solutions possess non-trivial topology one has to calculate value of the pertinent topological invariant. In our case static field configurations, which can be viewed as maps from S^3 into S^2 , are classified by so-called Hopf index $Q_H = \pi_3(S^2)$. One should notice that because of the fact that we deal with N fields thus N different topological Hopf charges $Q_H^{(i)}$ have to be introduced.

We define two additional functions

$$g_1^2 = \cosh \eta - \sqrt{\frac{n^2}{m^2} + \sinh^2 \eta} \quad (35)$$

and

$$g_2^2 = \sqrt{1 + \frac{m^2}{n^2} \sinh^2 \eta} - \cosh \eta. \quad (36)$$

Then, for every field we introduce four objects

$$\Phi_{\begin{pmatrix} 1 \\ 2 \end{pmatrix}}^{(i)} = \left(\frac{g_1}{\sqrt{g_1^2 + g_2^2}} \right) \times \begin{pmatrix} \cos m_i \xi \\ \sin m_i \xi \end{pmatrix} \quad (37)$$

and

$$\Phi_{\begin{pmatrix} 3 \\ 4 \end{pmatrix}}^{(i)} = \left(\frac{g_2}{\sqrt{g_1^2 + g_2^2}} \right) \times \begin{pmatrix} \cos n_i \phi \\ -\sin n_i \phi \end{pmatrix}. \quad (38)$$

They are related with the i -th unit vector field via the standard relations $\vec{n}_i = Z^{\dagger i} \vec{\sigma} Z^i$, where $\vec{\sigma}$ are Pauli matrices and

$$Z^i = \begin{pmatrix} Z_1^i \\ Z_2^i \end{pmatrix}, \quad Z^{i\dagger} = (Z_1^{i*}, Z_2^{i*}) \quad (39)$$

and

$$Z_1^i = \Phi_1^{(i)} + i\Phi_2^{(i)}, \quad Z_2^i = \Phi_3^{(i)} + i\Phi_4^{(i)}. \quad (40)$$

Using these objects one can construct N Abelian vector fields which form the previously introduced antisymmetric field tensors $H_{kl}^{(i)} = \partial_k A_l^{(i)} - \partial_l A_k^{(i)}$. In fact one finds that

$$A_k^{(i)} = \frac{i}{2} (Z^{i\dagger} \partial_k Z^i - \partial_k Z^{i\dagger} Z^i). \quad (41)$$

Then, using the standard expression for the Hopf index

$$Q_H^{(i)} = \frac{1}{4\pi^2} \int d^3x \vec{A}^{(i)} \cdot \vec{B}^{(i)}, \quad (42)$$

we arrive at the following result

$$Q_H^{(i)} = \frac{n_i m_i}{2} \left[((\Phi_1^{(i)})^2 + (\Phi_2^{(i)})^2)^2 - ((\Phi_3^{(i)})^2 + (\Phi_4^{(i)})^2)^2 \right]_0^\infty = -n_i m_i. \quad (43)$$

In other words our soliton solutions are classified by set of N Hopf charges. Every kind of the unit vector field carries its own topological charge which is unique determined by the parameters m_i, n_i . Now, one could ask about generalization of Vakulenko-Kapitansky inequality for N vector fields. Knowing that $\sqrt{1+|q|}\sqrt{q} \geq \sqrt{2}|q|^{3/4}$ we obtain the generalized relation

$$E \geq (2\pi)^2 4 \cdot 2^{\frac{3}{4}} \prod_{i=1}^N Q_{(i)}^{\alpha_i}, \quad (44)$$

where $\sum_{i=1}^N \alpha_i = \frac{3}{4}$. If all topological numbers are equal $Q_{(i)} = Q$, $i = 1..N$ we reproduce the standard Vakulenko-Kapitansky formula.

4 Conclusions

In the present work a generalized Aratyn-Ferreira-Zimerman model has been analyzed. This model consists of N unit vector fields and possesses N independent $O(3)$ symmetries. We have proved that this model is integrable in the sense that infinite number of the conserved currents appears. Precisely speaking, one can find a set of N infinite families of the currents.

We have also shown that, as in the case of the soliton theories in $(2+1)$ dimensions, this property leads to the existence of non-trivial topological field configurations. Our soliton solutions are described by shape functions depending on η coordinate and angular terms characterized by set of two integer parameters m_i and n_i . The total energy has been also found. In general, it is given by the integral (20). The exact form of the shape functions has been obtained in the limit of the constant ratio of the Ansatz parameters $n_i^2/m_i^2 = q^2 = \text{const}$. Then the shape functions take form of the well-know Aratyn-Ferreira-Zimerman solution. However, obtained solutions are much more general due to the angular part which for each of the unit vector field can be different. Because of that the topological behavior of the solutions is changed - now we have N independent topological Hopf indexes. In other words we have obtained multi-soliton configurations classified by N topological numbers $Q_H^{(i)}$, where each of the vector field carries different topological

charge. These configurations consist of N toroidal solitons of the different types.

It should be stressed that all properties of the solutions (topological charge, total energy), at least in the $q_i^2 = \text{const.}$ case, are determined only by value of the parameters m_i, n_i . For example, behavior of the shape functions f_i depends only on the ratio between them whereas number of the fields does not affect the shape functions. Regardless of the particular number of the fields N , all functions f_i possess identical form. On the other hand, number of the vector fields defines the topological content of the model. Similarly, particular values of the parameters α_i of the discussed model also do not inflect on the shape of solitons as well as on their topological charges.

In our opinion, the most interesting result is the generalized Vakulenko-Kapitanski inequality. As we see this formula can be divided into two parts. The first part contains only numbers and seems to be universal i.e. Lagrangian independent. In fact, it is identical as in the simple one vector field case. The second part, which includes the Hopf indexes, appears to be strongly sensitive to the form of the model. To show it more precisely one can consider a simple sum of N standard Aratyn-Ferriera-Zimmerman Lagrangians

$$\mathcal{L} = \sum_{i=1}^N (H_{\mu\nu}^{(i)2})^{3/4}$$

i.e. a model with the noninteracting unit vector fields. One can immediately show that the pertinent Vakuleko-Kapitanski formula reads

$$E \geq (2\pi)^2 4 \cdot 2^{\frac{3}{4}} \sum_{i=1}^N |Q^{(i)}|^{\frac{3}{4}}.$$

We see that the second part of the inequality reflects a form of the Lagrangian. However, a common feature is shared by both inequalities - there is an universal scaling property. Namely, if we transform all topological charges $Q^{(i)} \rightarrow \lambda Q^{(i)}$ then the total energy will scale as $E \rightarrow \lambda^{\frac{3}{4}} E$.

Let us now analyze the physical meaning of the parameters α_i . In spite of the fact that the form of the solution, at least in the $q_i^2 = \text{const.}$ regime, does not depend on their values, they play crucial role in the interaction of the toroidal solitons. Indeed, interaction between hopfions in the proposed model is governed by the values of α_i and it can be attractive or repulsive. Quite interesting hopfions can even do not interact at all. To be more precisely we will consider two hopfions with the following topological charges $Q_a = (Q_a^{(1)}, \dots, Q_a^{(N)})$, $a = 1, 2$. It follows from the formula (44) that if $\alpha_i < 1$ then i -th hopfions built of the i -th vector field should attract each other whereas for $\alpha_i > 1$ this interaction will be repulsive. In the very special

case where $\alpha_i = 1$, they do not interact at all. It resembles the Bogomolny limit in the standard soliton systems. As there is only one constrain on the parameters α_i (2), all three situations can be realized. Thus, in general, one can deal with a model where every possible type of interaction occurs. As we have mentioned it above, the analytical solutions have been obtained only in the simplest situation i. e. for $q_i^2 = \text{const.}$. It constrains also the validity of the generalized Vakulenko-Kapitanski inequality. Thus one should investigate more complicated cases with $q_i^2 \neq q_j^2$ as well. We would like to address this issue in our next paper.

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